

THE EXPONENTIAL MAP FOR REPRESENTATIONS OF $U_{p,q}(gl(2))^\dagger$

JORIS VAN DER JEUGT[‡]

*Department of Applied Mathematics and Computer Science,
University of Ghent, Krijgslaan 281-S9, B-9000 Gent, Belgium*

RAMASWAMY JAGANNATHAN[§]

*The Institute of Mathematical Sciences,
Madras - 600113, India*

For the quantum group $GL_{p,q}(2)$ and the corresponding quantum algebra $U_{p,q}(gl(2))$ Fronsdal and Galindo [1] explicitly constructed the so-called universal T -matrix. In a previous paper [2] we showed how this universal T -matrix can be used to exponentiate representations from the quantum algebra to get representations (left comodules) for the quantum group. Here, further properties of the universal T -matrix are illustrated. In particular, it is shown how to obtain comodules of the quantum algebra by exponentiating modules of the quantum group. Also the relation with the universal R -matrix is discussed.

1 Introduction and notation

Quantum groups may be studied in two different ways : the Drinfel'd-Jimbo approach from the point of view of integrable theories, and the Woronowicz-Manin picture from the point of view of pseudo-groups. Between these two approaches there is a duality relation [3,4]. Concretely, it means one can give dual bases for the “quantum group” on the one side (e.g. $GL_q(n)$) and for the “quantum algebra” on the other side (e.g. $U_q(gl(n))$), as dual Hopf algebras. An explicit example of such a dual basis was given by Fronsdal and Galindo [1] for the two-parameter quantum group $GL_{p,q}(2)$ and its dual $U_{p,q}(gl(2))$. This particular duality had been discussed before [5–7], but now Fronsdal and Galindo used it to construct the universal T -matrix (also called the canonical element [8]) yielding the interpretation as an exponential mapping from the quantum algebra to the quantum group. This work has been extended to other cases [9–11]. Here and in [2] we show that this exponential mapping can be applied to representations. In particular, we show how

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[‡]Senior Research Associate of N.F.W.O. (National Fund for Scientific Research of Belgium);
E-mail : Joris.VanderJeugt@rug.ac.be

[§]E-mail : jagan@imsc.ernet.in

left modules of the quantum algebra give rise to left comodules of the quantum group. Using the dual pairing this also leads to left modules of the quantum group and the related left comodules of the quantum algebra.

For the quantum group $GL_{p,q}(2)$ the defining T -matrix, or the defining representation matrix, is specified by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1)$$

with the commutation relations in terms of two parameters p and q :

$$\begin{aligned} ab &= qba, & cd &= qdc, & ac &= pca, & bd &= pdb, \\ bc &= (p/q)cb, & ad - da &= (q - p^{-1})bc, \end{aligned} \quad (2)$$

following from the RTT -relation $RT_1T_2 = T_2T_1R$, with $T_1 = T \otimes \mathbb{1}$, $T_2 = \mathbb{1} \otimes T$ and

$$R = Q^{\frac{1}{2}} \begin{pmatrix} Q^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & Q^{-1} - Q & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & Q^{-1} \end{pmatrix}, \quad Q = \sqrt{pq}, \quad \lambda = \sqrt{p/q}. \quad (3)$$

The coproduct and counit are easily defined in terms of T , e.g. $\Delta(T) = T \dot{\otimes} T$, and an antipode can be defined making $A_{p,q}(GL(2))$, the algebra of functions on $GL_{p,q}(2)$, into a Hopf algebra. For the definition of the antipode, $\mathcal{D} = ad - qbc = ad - pcb$, the quantum determinant of T , satisfying $\Delta(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}$, should be invertible. The noncentral \mathcal{D} satisfies

$$\mathcal{D}a = a\mathcal{D}, \quad \mathcal{D}b = \lambda^{-2}b\mathcal{D}, \quad \mathcal{D}c = \lambda^2c\mathcal{D}, \quad \mathcal{D}d = d\mathcal{D}. \quad (4)$$

The algebra \mathcal{U} dual to $\mathcal{A} = A_{p,q}(GL(2))$, namely $U_{p,q}(gl(2))$, is generated by the elements $\{J_0, J_{\pm}, Z\}$ subject to the relations

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = [2J_0]_Q, \quad [Z, \cdot] = 0, \quad (5)$$

where, as usual, $[x]_t = (t^x - t^{-x})/(t - t^{-1})$. We recall here the form of the coproduct :

$$\begin{aligned} \Delta(J_{\pm}) &= J_{\pm} \otimes Q^{-J_0} \lambda^{\pm Z} + Q^{J_0} \lambda^{\mp Z} \otimes J_{\pm}, \\ \Delta(J_0) &= J_0 \otimes \mathbb{1} + \mathbb{1} \otimes J_0, \quad \Delta(Z) = Z \otimes \mathbb{1} + \mathbb{1} \otimes Z; \end{aligned} \quad (6)$$

the remaining functions (counit, antipode) can be found elsewhere [6,7]. Apart from Δ , there exists also the opposite coproduct $\Delta' = \sigma\Delta$, with $\sigma(u \otimes v) = v \otimes u$. For a quasitriangular Hopf algebra such as $\mathcal{U} = U_{p,q}(gl(2))$ these two coproducts are related by $\Delta'\mathcal{R} = \mathcal{R}\Delta$ through an element $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ called the universal R -matrix and explicitly given by [12]

$$\begin{aligned} \mathcal{R} &= Q^{-2(J_0 \otimes J_0)} \lambda^{2(Z \otimes J_0 - J_0 \otimes Z)} \\ &\times \sum_{n=0}^{\infty} \frac{(1 - Q^2)^n}{[n]!} Q^{-\frac{1}{2}n(n-1)} (Q^{-J_0} \lambda^Z J_+ \otimes Q^{J_0} \lambda^Z J_-)^n \end{aligned} \quad (7)$$

where $[n] = [n]_Q$ and $[n]! = [n][n-1] \dots [2][1]$.

2 The universal T -matrix

The exponential map from $\mathcal{U} = U_{p,q}(gl(2))$ to $A_{p,q}(GL(2))$, *à la* Fronsda and Galindo [1] is defined in terms of the so-called universal T -matrix \mathcal{T} . First, one has to construct a dual basis for the dual Hopf algebras \mathcal{A} and \mathcal{U} . The fact that \mathcal{A} and \mathcal{U} are dual means that product, coproduct, unit, counit and antipode in \mathcal{A} and \mathcal{U} are related as follows by the dual pairing :

$$\begin{aligned} \langle XY, x \rangle &= \langle X \otimes Y, \Delta x \rangle, & \langle \Delta X, x \otimes y \rangle &= \langle X, xy \rangle, & \langle 1, x \rangle &= \epsilon(x), \\ \epsilon(X) &= \langle X, 1 \rangle, & \langle SX, x \rangle &= \langle X, Sx \rangle, & X, Y &\in \mathcal{U}, \quad x, y \in \mathcal{A}. \end{aligned} \quad (8)$$

A dual basis then consists of a basis $\{x^A\}$ for \mathcal{A} and $\{X_A\}$ for \mathcal{U} such that the pairing is equal to $\langle X_A, x^B \rangle = \delta_A^B$ (Kronecker δ). As a consequence, the structure constants for the product (resp. coproduct) in \mathcal{A} become the structure constants for the coproduct (resp. product) in \mathcal{U} :

$$\begin{aligned} x^B x^C &= \sum_A h_A^{BC} x^A, & \Delta(x^C) &= \sum_{A,B} f_{AB}^C x^A \otimes x^B, \\ X_A X_B &= \sum_C f_{AB}^C X_C, & \Delta(X_A) &= \sum_{B,C} h_A^{BC} X_B \otimes X_C. \end{aligned} \quad (9)$$

The universal T -matrix, \mathcal{T} , can now be defined as an element of $\mathcal{A} \otimes \mathcal{U}$:

$$\mathcal{T} = \sum_A x^A \otimes X_A. \quad (10)$$

Using the duality properties, it is easy to show that \mathcal{T} satisfies the following properties :

$$\begin{aligned} (\Delta \otimes \mathbb{1})(\mathcal{T}) &= \sum_A \Delta(x^A) \otimes X_A = \mathcal{T}_{13} \mathcal{T}_{23}, \\ (\mathbb{1} \otimes \Delta)(\mathcal{T}) &= \sum_A x^A \otimes \Delta(X_A) = \mathcal{T}_{12} \mathcal{T}_{13}, \end{aligned} \quad (11)$$

where, e.g. \mathcal{T}_{13} stands for $\sum_A x^A \otimes 1 \otimes X_A$.

What is now this dual basis and the corresponding universal T -matrix for the case $\mathcal{U} = U_{p,q}(gl(2))$ and $\mathcal{A} = A_{p,q}(GL(2))$? To this end, the generators of $U_{p,q}(gl(2))$ are redefined as follows :

$$\hat{J}_+ = J_+ Q^{-(J_0 + \frac{1}{2})} \lambda^{Z - \frac{1}{2}}, \quad \hat{J}_- = Q^{(J_0 + \frac{1}{2})} \lambda^{Z - \frac{1}{2}} J_-, \quad \hat{J}_0 = J_0, \quad \hat{Z} = Z, \quad (12)$$

with, of course, the corresponding products and coproducts. For $\mathcal{A} = A_{p,q}(GL(2))$, with the extra assumption that a is invertible, the new variables $\{\alpha, \beta, \gamma, \delta\}$ defined through

$$a = e^\alpha, \quad \beta = a^{-1}b, \quad \gamma = ca^{-1}, \quad d = ca^{-1}b + e^{-\delta}. \quad (13)$$

form a Lie algebra with structure constants depending upon $\rho = \ln(Q)$ and $\theta = \ln(\lambda)$:

$$\begin{aligned} [\alpha, \beta] &= (\rho - \theta)\beta, & [\alpha, \gamma] &= (\rho + \theta)\gamma, \\ [\delta, \beta] &= (\rho + \theta)\beta, & [\delta, \gamma] &= (\rho - \theta)\gamma, \\ [\alpha, \delta] &= 0, & [\beta, \gamma] &= 0. \end{aligned} \quad (14)$$

With these new generating elements, Fronsdal and Galindo [1] obtained an explicit dual basis for $A_{p,q}(GL(2))$ and $U_{p,q}(gl(2))$ given by :

$$\begin{aligned} x^A &= \gamma^{a_1} \alpha^{a_2} \delta^{a_3} \beta^{a_4}, \\ X_A &= \frac{Q^{\frac{1}{2}a_1(a_1-1)} \hat{J}_-^{a_1}}{[a_1]!} \frac{(\hat{J}_0 + \hat{Z})^{a_2}}{a_2!} \frac{(\hat{J}_0 - \hat{Z})^{a_3}}{a_3!} \frac{Q^{-\frac{1}{2}a_4(a_4-1)} \hat{J}_+^{a_4}}{[a_4]!}, \\ & \quad a_1, a_2, a_3, a_4 = 0, 1, 2, \dots \end{aligned} \quad (15)$$

Then, with the definition of a basic exponential function with parameter t^2 as

$$\mathcal{E}_{t^2}(x) = \sum_{n=0}^{\infty} \frac{t^{-\frac{1}{2}n(n-1)}}{[n]_t!} x^n, \quad (16)$$

the universal T -matrix (10) can be written as

$$\mathcal{T} = \mathcal{E}_{Q^{-2}}(\gamma \otimes \hat{J}_-) e^{\alpha \otimes (\hat{J}_0 + \hat{Z}) + \delta \otimes (\hat{J}_0 - \hat{Z})} \mathcal{E}_{Q^2}(\beta \otimes \hat{J}_+). \quad (17)$$

3 Applications

The first application was discussed in [2]. There, it was shown that (17) can be used to exponentiate representations of $U_{p,q}(gl(2))$ to representations of $GL_{p,q}(2)$. For generic parameters, the finite dimensional irreducible representations of $U_{p,q}(gl(2))$ are labelled by two numbers, the ‘spin’ j (integer or half-integer) and the Z -eigenvalue z . The explicit matrix elements of the generating elements in this $(2j+1) \times (2j+1)$ representation $(j; z)$ are given by

$$\begin{aligned} (J_{\pm})_{mk} &= ([j \pm m][j + 1 \mp m])^{1/2} \delta_{m, k \pm 1}, \\ (J_0)_{mk} &= m \delta_{mk}, & (Z)_{mk} &= z \delta_{mk}, \\ & \quad m, k = j, j-1, \dots, -(j-1), -j. \end{aligned} \quad (18)$$

Exponentiating this representation means substituting these matrices in the formula (17). This gives rise to a matrix of order $2j+1$ with elements depending upon α, β, γ and δ , or using the relations (13) eventually upon a, b, c and d . This matrix, denoted by $T^{(j; z)}$, was calculated explicitly in [2], and its elements are given as follows :

$$T_{mk}^{(j; z)} = \mathcal{D}^{z-j} Q^{-(m-k)(2j-m+k)/2} \lambda^{-(m-k)(2j-2z-m-k)}$$

$$\begin{aligned}
& \times ([j+m]![j-m]![j+k]![j-k]!)^{1/2} \\
& \times \sum_s Q^{-s(2j-m+k-s)} \lambda^{-s(m-k+s)} \\
& \times \frac{a^{j+k-s}}{[j+k-s]!} \frac{b^{m-k+s}}{[m-k+s]!} \frac{c^s}{[s]!} \frac{d^{j-m-s}}{[j-m-s]!}. \quad (19)
\end{aligned}$$

In the limit $p = q$, or $Q = q$ and $\lambda = 1$, and $\mathcal{D} = 1$, $GL_{p,q}(2)$ becomes $SL_q(2)$. Then, the above matrices coincide with the representations of $SL_q(2)$ as obtained earlier in different approaches [13–19].

As an example, for $(j; z) = (1/2; 1/2)$, the matrix $T^{(j;z)}$ simply becomes the defining T -matrix given in (1). For $(j; z) = (1; 1/2)$, one finds

$$T^{(1;1/2)} = \mathcal{D}^{-1/2} \begin{pmatrix} a^2 & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} ab & \lambda^{-1} b^2 \\ [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} ac & ad + Q^{-1} \lambda^{-1} bc & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} \lambda^{-1} bd \\ \lambda c^2 & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} \lambda cd & d^2 \end{pmatrix}. \quad (20)$$

Because of the properties of the universal T -matrix \mathcal{T} , the matrices $T^{(j;z)}$ satisfy the following properties :

$$\Delta(T_{kl}^{(j;z)}) = \sum_{i=-j}^j T_{ki}^{(j;z)} \otimes T_{il}^{(j;z)}, \quad \text{i.e.} \quad \Delta(T^{(j;z)}) = T^{(j;z)} \dot{\otimes} T^{(j;z)}. \quad (21)$$

In particular, if the elements of a column of the matrix $T^{(j;z)}$ are denoted by v_l , then $\Delta(v_l) = \sum_i T_{li}^{(j;z)} \otimes v_i$. This means that we are dealing with a left \mathcal{A} -comodule.

For the second application, we shall take $\mathcal{A} = A_q(SL(2))$ and $\mathcal{U} = U_q(sl(2))$ in which case $p = q$ (or, $Q = q$, $\lambda = 1$) and $\mathcal{D} = 1$. It is well known that the relations (and other properties such as Δ , ϵ and S) for \mathcal{U} can also be described in terms of the standard L -matrices L^\pm , together with the standard R -matrix (3) :

$$RL_2^\pm L_1^\pm = L_1^\pm L_2^\pm R, \quad RL_2^+ L_1^- = L_1^- L_2^+ R. \quad (22)$$

Explicitly, we have

$$L^+ = \begin{pmatrix} q^{-J_0} & 0 \\ q^{1/2}(q^{-1} - q)J_+ & q^{J_0} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{J_0} & q^{-1/2}(q - q^{-1})J_- \\ 0 & q^{-J_0} \end{pmatrix}. \quad (23)$$

It is known that these matrices satisfy the defining relations for $SL_{q^{-1}}(2)$. Hence, changing q to q^{-1} in the above expressions leads to two maps π^- and π^+ which are Hopf algebra homomorphisms :

$$\pi^- : \begin{cases} a \rightarrow q^{J_0} \\ b \rightarrow 0 \\ c \rightarrow q^{-1/2}(q - q^{-1})J_+ \\ d \rightarrow q^{-J_0} \end{cases} \quad \pi^+ : \begin{cases} a \rightarrow q^{-J_0} \\ b \rightarrow q^{1/2}(q^{-1} - q)J_- \\ c \rightarrow 0 \\ d \rightarrow q^{J_0} \end{cases}. \quad (24)$$

Given the $(2j+1)$ -dimensional representation $\Gamma^{(j)}$ of $U_q(sl(2))$ (see (18) with $Q = q$ and $\lambda = 1$), applying $\Gamma^{(j)} \circ \pi^\pm$ yields $(2j+1)$ -dimensional representations for a, b, c, d (left modules for $SL_q(2)$), and thus also matrix representations for $\alpha, \beta, \gamma, \delta$. These representations for a, b, c, d coincide with the ones obtained from the universal R -matrix as follows. Let $R^{(1/2) \otimes (j)}$ be the matrix obtained from the universal R -matrix by restriction to the representations specified, $(R^{(1/2) \otimes (j)})^{-1}$ its inverse, and $T = (t_{ik})$ the standard T -matrix (1). Then

$$(\Gamma^{(j)} \circ \pi^+)(t_{ik})_{lm} = R_{il, km}^{(1/2) \otimes (j)}, \quad (\Gamma^{(j)} \circ \pi^-)(t_{ik})_{lm} = (R^{(1/2) \otimes (j)})_{li, mk}^{-1}. \quad (25)$$

Substituting these representations in \mathcal{T} as follows

$$L^{+(j)} = \left((\Gamma^{(j)} \circ \pi^+) \otimes \mathbb{1} \right) (\mathcal{T}), \quad L^{-(j)} = \left((\Gamma^{(j)} \circ \pi^-) \otimes \mathbb{1} \right) (\mathcal{T}), \quad (26)$$

one obtains $(2j+1) \times (2j+1)$ matrices with elements that are polynomials of degree $2j$ in J_\pm and $q^{\pm J_0}$. In particular, one finds that $L^{\pm(1/2)} = L^\pm$. For $j = 1$,

$$L^{+(1)} = \begin{pmatrix} q^{-2J_0} & 0 & 0 \\ (1-q^2)\sqrt{[2]}q^{-J_0}J_+ & 1 & 0 \\ q^{-1}(1-q^2)^2J_+^2 & q^{-1}(1-q^2)\sqrt{[2]}q^{J_0}J_+ & q^{2J_0} \end{pmatrix}, \quad (27)$$

$$L^{-(1)} = \begin{pmatrix} q^{2J_0} & q(1-q^{-2})\sqrt{[2]}q^{J_0}J_- & q(1-q^{-2})^2J_-^2 \\ 0 & 1 & (1-q^{-2})\sqrt{[2]}q^{-J_0}J_- \\ 0 & 0 & q^{-2J_0} \end{pmatrix}. \quad (28)$$

The matrices $L^{\pm(j)}$ are the spin j analogs of the standard L -matrices. They satisfy the properties

$$R^{(j) \otimes (j)} L_2^{\pm(j)} L_1^{\pm(j)} = L_1^{\pm(j)} L_2^{\pm(j)} R^{(j) \otimes (j)}, \quad (29)$$

$$R^{(j) \otimes (j)} L_2^{+(j)} L_1^{-(j)} = L_1^{-(j)} L_2^{+(j)} R^{(j) \otimes (j)}, \quad (30)$$

$$\Delta(L^{\pm(j)}) = L^{\pm(j)} \dot{\otimes} L^{\pm(j)},$$

and thus they constitute the $(2j+1)$ -dimensional comodules of $U_q(sl(2))$.

As a third and last application, we write \mathcal{T} in a different form :

$$\mathcal{T}' = \sum_A X_A \otimes x^A \in \mathcal{U} \otimes \mathcal{A}. \quad (31)$$

Then

$$(\mathbb{1} \otimes \pi^+) \mathcal{T}' = \mathcal{R}, \quad (\mathbb{1} \otimes \pi^-) \mathcal{T}' = \mathcal{R}. \quad (32)$$

This relationship between the universal T -matrix and the universal R -matrix has also been considered by Fronsdaal [20].

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